

DISCRETE MATHEMATICS: COMBINATORICS AND GRAPH THEORY

Practice Exam 1

Instructions. Solve any 5 questions. Write neatly and show your work to receive full credit. Note that this is a sample exam and while it bears some similarity with the real exam, the two are not isomorphic.

1. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.

(a) $\sum_{i=1}^n 2^{i-1} = \sum_{i=0}^{n-1} 2^i = 2^n - 1$

(b) $\sum_{i=1}^n i(2^i) = 2 + (n-1)2^{n+1}$

(c) $\sum_{i=1}^n i(i!) = (n+1)! - 1$

2. Prove the following:

(a) The sum of any three consecutive integers is divisible by 3.

(b) For all integers a , b , and c , if a divides b and a divides c then a divides $b + c$.

(c) Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 1$. Hint: use proof by contradiction.

3. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.

(a) For all $n \in \mathbb{Z}^+$, $n > 3 \Rightarrow 2^n < n!$

(b) For all $n \in \mathbb{Z}^+$, $n > 4 \Rightarrow n^2 < 2^n$

(c) For all $n \in \mathbb{N}$, $3|(7^n - 4)$

4. Verify whether the following functions define bijections.

(a) Let $m \neq 0$ and b be real numbers. Is the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = mx + b$ (i) injective (ii) surjective and (iii) bijective? Prove or disprove.

(b) Let $S = \{x \in \mathbb{R} : x \neq 1\}$. Is the function $f : S \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x+1}{x-1}$ (i) injective (ii) surjective and (iii) bijective? Prove or disprove.

5. Use the Euclidean Algorithm to find the greatest common divisor of 44 and 17. Use the Extended Euclidean Algorithm to find the Bezout coefficients x, y and all integer solutions to the equation $44x + 17y = \gcd(44, 17)$.

6. The Fermat numbers are defined as $f(n) = 2^{2^n} + 1$. Prove that for all $n \geq 2$, $f_{n+1} = f_0 \times f_1 \times \cdots \times f_n + 2$.